Recall the **Intermediate Value Theorem**: if $f : [a, b] \to \mathbb{R}$ is continuous and $f(a) < f(b)$, then for every $d \in [f(a), f(b)]$, there is some $c \in [a, b]$ such that $f(c) = d$. In other words, f attains all values between $f(a)$ and $f(b)$.

Problem 1
Suppose f and g are continuous on $[a, b]$ and that $f(a) < g(a)$, but $f(b) > g(b)$. Prove that $f(x) = g(x)$ for some x in $[a, b]$.

Problem 2

1. Let $f : [a, b] \to \mathbb{R}$ be continuous. Suppose that f only takes on integer values; in other words, $f(x) \in \mathbb{Z}$ for all $x \in [a, b]$. Using contradiction and intermediate value theorem, show that f is constant.
2. Can we conclude the same thing if instead we replaced the hypothesis $f(x) \in \mathbb{Z}$ for all $x \in [a, b]$ with $f(x) \in \mathbb{Q}$ for all $x \in [a, b]$?

Problem 3
Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and increasing.

1. Show that f need not be surjective.
2. Suppose furthermore that
$$\lim_{x \to -\infty} f(x) = -\infty$$
and
$$\lim_{x \to \infty} f(x) = \infty.$$
Show that f is surjective, using the intermediate value theorem.

Problem 4
Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 3x^3 - 6x - 1$. Show that the function f has at least two roots, without explicitly finding the roots.

Problem 5

1. Suppose that f is a continuous function on $[0, 1]$ and that $f(x) \in [0, 1]$ for each x. Prove that $f(x) = 1 - x$ for some number x.
2. Suppose f is as in the previous subproblem, and g is continuous on $[0, 1]$ with $g(0) = 1$, $g(1) = 0$. Show $f(x) = g(x)$ for some x using a similar procedure.